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2008 J. Phys. A: Math. Theor. 41 045007

(http://iopscience.iop.org/1751-8121/41/4/045007)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 045007 (9pp) doi:10.1088/1751-8113/41/4/045007

Bose–Einstein condensation of a relativistic q-deformed Bose gas

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Received 12 October 2007, in final form 5 December 2007
Published 15 January 2008
Online at stacks.iop.org/JPhysA/41/045007

Abstract

The q-deformed Bose–Einstein distribution derived from the q-boson algebra is used to study the low-temperature behavior of an ideal q-deformed Bose gas with relativistic energy spectrum. The effects of q-deformation on the properties relative to Bose–Einstein condensation (BEC) are discussed. It is shown that q-deformation leads to some novel characteristics different from those of an original Bose gas, which include the criteria on the occurrence of BEC, critical temperature and jump of heat capacity at the critical point. The results obtained here provide a unified description for the properties of q-deformed Bose systems from the nonrelativistic case to the ultrarelativistic limit, so that some important conclusions in the literature are included in this paper.

PACS numbers: 05.30.Jp, 03.75.Hh, 03.30.+p

1. Introduction

The theory of q-bosons or q-oscillators, which were first introduced by Sklyanin and Kulish in their studies of solvable models and the related Yang–Baxter equations [1, 2] and developed by many other researchers [3, 4], has become a topic of great interest in the past few years. Its possible applications have been found in a wide range of areas, such as anyon physics [5, 6], vertex models [7] and quantum mechanics in discontinuous spacetime [8], etc. Recent investigations in the theory of q-bosons have provided much insight into the mathematical development as well as the q-deformed thermodynamics [9–23].

Q-deformed boson algebra is realized by defining the q-deformed creation operator $\hat{a}^+$ and annihilation operator $\hat{a}^-$ which satisfy the commutation relations [12]

$$[\hat{n},\ \hat{a}^+] = \hat{a}^+, \qquad [\hat{n},\ \hat{a}] = -\hat{a} \tag{1}$$

and the nonlinear relations

$$\hat{a}^+\hat{a} = [\hat{n}], \qquad \hat{a}\hat{a}^+ = [\hat{n}+1], \tag{2}$$

where $\hat{n}$ is the number operator, $[x]$ denotes

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}, \tag{3}$$

and $q \in R^+$ is the deformation parameter.

For an ideal q-deformed boson system, the Hamitonian is given by

$$\hat{H} = \sum_k (\varepsilon_k - \mu)\hat{n}_k, \tag{4}$$

and the mean value of the occupation number $f_{k,q}$ is defined as [12]

$$[f_{k,q}] = \frac{\mathrm{tr}\{\exp(-\beta\hat{H})[\hat{n}_k]\}}{\mathrm{tr}\{\exp(-\beta\hat{H})\}}, \tag{5}$$

where k is the state label, $\hat{n}_k$ and ε_k are, respectively, the number operator and energy of state k, μ is the chemical potential, $\beta = 1/k_B T$, k_B is the Boltzmann constant and T is the temperature of the system. According to equations (1)–(5), one can derive the statistical distribution of q-deformed bosons as

$$f_{k,q} = \frac{1}{2\ln q}\ln\frac{\exp[\beta(\varepsilon_k - \mu)] - q^{-1}}{\exp[\beta(\varepsilon_k - \mu)] - q}. \tag{6}$$

It is easy to prove that $f_{k,q}$ possesses the following properties:

(1) When $q \to 1$, equation (6) becomes

$$f_{k,1} = \frac{1}{\exp[\beta(\varepsilon_k - \mu)] - 1}, \tag{7}$$

which is just the standard Bose–Einstein distribution. This means that q-deformed bosons will be the same as ordinary bosons when $q \to 1$.

(2) $f_{k,q} = f_{k,1/q}$, which implies that the properties of q-deformed bosons will remain unchanged under the transformation of $q \leftrightarrow 1/q$. Thus, we shall restrict the following discussions to the case of $q \geqslant 1$.

Using equation (6), some authors studied the properties of ideal q-bosons with the nonrelativistic energy spectrum $\varepsilon(p) = p^2/2m$ [12, 18], where p and m are, respectively, the momentum and rest mass of a particle. The authors in [20] extended the works and explored the low-temperature behavior of an ideal q-bosons with more general energy spectrum $\varepsilon(p) = ap^s$ (a and s are positive constants), which includes the nonrelativistic and ultrarelativistic energy spectrums for the special cases of $s = 2$ and $s = 1$, respectively. For some special systems, however, the particles generally possess the relativistic energy spectrum as

$$\varepsilon(p) = \sqrt{p^2c^2 + m^2c^4} \tag{8}$$

and it is necessary to further discuss the behavior of q-bosons from the nonrelativistic case to the ultrarelativistic limit.

In the present paper, we will deal with the problems relative to the relativistic Bose–Einstein condensation (BEC) of an ideal q-deformed Bose gas in any dimensional space. The paper is organized as follows: in section 2, we give a theoretical evaluation of some important parameters, such as the critical temperature, ground-state fraction, total energy and heat capacity of the relativistic q-deformed Bose gas. In section 3, we further discuss the low-temperature behavior of the system and pay close attention to the effects of q-deformation on the characteristics of BEC. Finally, some important conclusions are summed up.

2. Theoretical evaluation

Consider an ideal relativistic q-deformed Bose gas confined in a D-dimensional rigid box with the length L each side. According to equation (6), the total number of particles and total energy of the system can be, respectively, expressed as

$$N = \sum_k \frac{1}{2\ln q} \ln \frac{\exp[\beta(\varepsilon_k - \mu)] - q^{-1}}{\exp[\beta(\varepsilon_k - \mu)] - q} \tag{9}$$

and

$$E = \sum_k \frac{\varepsilon_k}{2\ln q} \ln \frac{\exp[\beta(\varepsilon_k - \mu)] - q^{-1}}{\exp[\beta(\varepsilon_k - \mu)] - q}. \tag{10}$$

Under the thermodynamic limit, ε_k and $\sum_k$ in the above equations can be, respectively, replaced by $\varepsilon(p)$ given by equation (8) and the integral over the phase space, i.e.,

$$\begin{aligned} N &= N_0 + \frac{1}{h^D}\int \frac{1}{2\ln q} \ln \frac{\exp[\beta(\sqrt{p^2c^2+m^2c^4} - \mu)] - q^{-1}}{\exp[\beta(\sqrt{p^2c^2+m^2c^4} - \mu)] - q} d^D\boldsymbol{r} d^D\boldsymbol{p} \\ &= N_0 + \frac{1}{h^D}\sum_{j=1}^{\infty} \frac{(q^j - q^{-j})}{2\ln q}\frac{\exp(j\beta\mu)}{j}\int \exp[-j\beta(\sqrt{p^2c^2+m^2c^4})] d^D\boldsymbol{r} d^D\boldsymbol{p} \\ &= N_0 + \left(\frac{L}{\lambda_c}\right)^D \left(\frac{2\pi}{u}\right)^{D'-1}\left[\sum_{j=1}^{\infty}\frac{(q^j - q^{-j})}{\ln q}\frac{\exp(j\beta\mu)}{j^{D'}} K_{D'}(ju)\right] \end{aligned} \tag{11}$$

and

$$\begin{aligned} E &= N_0 mc^2 + \frac{1}{h^D}\int \frac{\sqrt{p^2c^2+m^2c^4}}{2\ln q} \ln \frac{\exp[\beta(\sqrt{p^2c^2+m^2c^4} - \mu)] - q^{-1}}{\exp[\beta(\sqrt{p^2c^2+m^2c^4} - \mu)] - q} d^D\boldsymbol{r} d^D\boldsymbol{p} \\ &= Nmc^2 + k_B T \left(\frac{L}{\lambda_c}\right)^D \left(\frac{2\pi}{u}\right)^{D'-1} \\ &\quad\times \sum_{j=1}^{\infty}\frac{(q^j - q^{-j})}{\ln q}\frac{\exp(j\beta\mu)}{j^{D'+1}}[(-ju-1)K_{D'}(ju) + ju K_{D'+1}(ju)], \end{aligned} \tag{12}$$

where $D' = (D+1)/2$, $u = \beta mc^2$, $\lambda_c = h/mc$, h is the Planck constant,

$$K_\nu(x) = \frac{\sqrt{\pi}}{\Gamma(\nu + 1/2)}\left(\frac{x}{2}\right)^\nu \int_0^\infty \exp(-x\cosh\theta)\sinh^{2\nu}\theta\, \mathrm{d}\theta \tag{13}$$

is the modified Bessel function, and

$$N_0 = \frac{1}{2\ln q}\ln\frac{\exp(u - \beta\mu) - q^{-1}}{\exp(u - \beta\mu) - q} \tag{14}$$

is the number of particles in the ground state. The ground-state occupation should be nonnegative, i.e., $N_0 \geqslant 0$. This requires that $\mu \leqslant mc^2 - k_B T \ln q$.

When $\mu \to mc^2 - k_B T \ln q$ and N_0 is still macroscopically negligible, BEC begins to occur in the system. According to equation (11), the critical temperature T_C of BEC can be determined by

$$N = \left(\frac{L}{\lambda_c}\right)^D \left(\frac{2\pi}{u_C}\right)^{D'-1}\left[\sum_{j=1}^{\infty}\frac{(1 - q^{-2j})}{\ln q}\frac{\exp(ju_C)}{j^{D'}} K_{D'}(ju_C)\right], \tag{15}$$

where $u_C = \beta_C mc^2$ and $\beta_C = 1/k_B T_C$.

When $T < T_C$, the ground state is macroscopically occupied. Using equations (11) and (15), one can find the ground-state fraction as

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_C}\right)^{D'-1} \frac{\sum_{j=1}^{\infty} \frac{(1-q^{-2j})\exp(ju)}{j^{D'}} K_{D'}(ju)}{\sum_{j=1}^{\infty} \frac{(1-q^{-2j})\exp(ju_C)}{j^{D'}} K_{D'}(ju_C)}. \tag{16}$$

According to equations (11) and (12) and the property of the modified Bessel function

$$x \frac{\mathrm{d} K_\nu(x)}{\mathrm{d}x} = \nu K_\nu(x) - x K_{\nu+1}(x), \tag{17}$$

one can calculate the heat capacity at the given number of particles and constant volume, i.e.,

$$\begin{aligned} C_{T>T_C} &= \left(\frac{\partial E}{\partial T}\right)_{N,L} = \left(\frac{\partial E}{\partial T}\right)_{N,L,\mu} + \left(\frac{\partial E}{\partial \mu}\right)_{N,L,T} \left(\frac{\partial \mu}{\partial T}\right)_{N,L} \\ &= N k_B \left\{ \frac{u \sum_{j=1}^{\infty} (q^j - q^{-j}) \frac{\exp(j\beta\mu)}{j^{D'}} \left[{(2+ju) K_{D'}(ju) - (3+2ju) K_{D'+1}(ju) \atop + ju K_{D'+2}(ju)} \right]}{\sum_j^{\infty} (q^j - q^{-j}) \frac{\exp(j\beta\mu)}{j^{D'}} K_{D'}(ju)} \right. \\ &\quad \left. - \frac{\left\{\sum_{j=1}^{\infty} (q^j - q^{-j}) \frac{\exp(j\beta\mu)}{j^{D'}} [(-1 - ju) K_{D'}(ju) + ju K_{D'+1}(ju)]\right\}^2}{\sum_{j=1}^{\infty} (q^j - q^{-j}) \frac{\exp(j\beta\mu)}{j^{D'}} K_{D'}(ju) \sum_{j=1}^{\infty} (q^j - q^{-j}) \frac{\exp(j\beta\mu)}{j^{D'-1}} K_{D'}(ju)} \right\} \end{aligned} \tag{18}$$

and

$$\begin{aligned} C_{T\leqslant T_C} &= \left(\frac{\partial E}{\partial T}\right)_{N,L} \\ &= N k_B \left(\frac{T}{T_C}\right)^{D'-1} \frac{u \sum_{j=1}^{\infty} (1 - q^{-2j}) \frac{\exp(ju)}{j^{D'}} \left[{(2+ju) K_{D'}(ju) - (3+2ju) K_{D'+1}(ju) \atop + ju K_{D'+2}(ju)} \right]}{\sum_j^{\infty} (1 - q^{-2j}) \frac{\exp(ju_C)}{j^{D'}} K_{D'}(ju_C)}. \end{aligned} \tag{19}$$

By using equations (18) and (19), the jump of the heat capacity between $T \to T_C^-$ and $T \to T_C^+$ is obtained as

$$\begin{aligned} \Delta C &\equiv C_{T=T_C^-} - C_{T=T_C^+} \\ &= N k_B \frac{\left\{\sum_{j=1}^{\infty} (1 - q^{-2j}) \frac{\exp(ju_C)}{j^{D'}} [(-1 - ju_C) K_{D'}(ju_C) + ju_C K_{D'+1}(ju_C)]\right\}^2}{\sum_{j=1}^{\infty} (1 - q^{-2j}) \frac{\exp(ju_C)}{j^{D'}} K_{D'}(ju_C) \sum_{j=1}^{\infty} (1 - q^{-2j}) \frac{\exp(ju_C)}{j^{D'-1}} K_{D'}(ju_C)}. \end{aligned} \tag{20}$$

Equations (11), (15), (16) and (18)–(20) provide a unified description for the properties of the q-deformed Bose gas from the nonrelativistic case to the ultrarelativistic limit.

When $u = \beta mc^2 \to \infty$ and $u_C = \beta_C mc^2 \to \infty$, by using the expansion of the modified Bessel function for a large argument

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}} \exp(-x) \left(1 + \frac{4\nu^2 - 1}{8x} + \frac{16\nu^4 - 40\nu^2 + 9}{128x^2} + \cdots\right), \tag{21}$$

equations (11), (15), (16) and (18)–(20) are, respectively, reduced to

$$N = N_0 + \left(\frac{L}{\lambda_{nr}}\right)^D g_{q,D/2+1}(z), \tag{22}$$

$$T_C = \frac{h^2}{2\pi m k_B L^2} \left[\frac{N}{\zeta_q(D/2+1)}\right]^{2/D}, \tag{23}$$

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_C}\right)^{D/2}, \tag{24}$$

$$C_{T>T_C} = Nk_B\left[\frac{D}{2}\left(\frac{D}{2}+1\right)\frac{g_{q,D/2+2}(z)}{g_{q,D/2+1}(z)} - \frac{D^2}{4}\frac{g_{q,D/2+1}(z)}{g_{q,D/2}(z)}\right], \tag{25}$$

$$C_{T\leqslant T_C} = Nk_B\frac{D}{2}\left(\frac{D}{2}+1\right)\frac{\zeta_q(D/2+2)}{\zeta_q(D/2+1)}\left(\frac{T}{T_C}\right)^{D/2}, \tag{26}$$

and

$$\Delta C = Nk_B\frac{D^2}{4}\frac{\zeta_q(D/2+1)}{\zeta_q(D/2)}, \tag{27}$$

where

$$g_{q,\nu}(x) = \frac{1}{2\ln q}\sum_{j=1}^{\infty}(q^j - q^{-j})\frac{x^j}{j^\nu} \tag{28}$$

is the expansion of the q-deformed Bose integral [20], $\zeta_q(\nu) = g_{q,\nu}(1/q)$ is the q-deformed Riemann Zeta function, $z = \exp[\beta(\mu - mc^2)]$ is the fugacity and $\lambda_{nr} = h/\sqrt{2\pi m k_B T}$ is the nonrelativistic thermal wavelength. Equations (22)–(27) represent the statistical properties of a nonrelativistic q-deformed Bose gas and are just the same as those obtained in [12, 18].

On the other hand, when $u = \beta mc^2 \to 0$ and $u_C = \beta_C mc^2 \to 0$, by using

$$K_\nu(x) \stackrel{x\to 0}{\longrightarrow} \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu \tag{29}$$

and the approximate conditions $\exp(ju) \approx 1$ and $\exp(ju_C) \approx 1$, equations (11), (15), (16) and (18)–(20) are, respectively, simplified as

$$N = N_0 + \left(\frac{L}{\lambda_{ur}}\right)^D g_{q,D+1}(z), \tag{30}$$

$$T_C = \frac{hc}{2k_B L}\left[\frac{N}{\pi^{(D-1)/2}\Gamma(D/2+1/2)\zeta_q(D+1)}\right]^{1/D}, \tag{31}$$

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_C}\right)^D, \tag{32}$$

$$C_{T>T_C} = Nk_B\left[D(D+1)\frac{g_{q,D+2}(z)}{g_{q,D+1}(z)} - D^2\frac{g_{q,D+1}(z)}{g_{q,D}(z)}\right], \tag{33}$$

$$C_{T\leqslant T_C} = Nk_B D(D+1)\frac{\zeta_q(D+2)}{\zeta_q(D+1)}\left(\frac{T}{T_C}\right)^D, \tag{34}$$

and

$$\Delta C = Nk_B D^2\frac{\zeta_q(D+1)}{\zeta_q(D)}, \tag{35}$$

where

$$\lambda_{ur} = \frac{hc}{2k_B T}\left[\frac{1}{\pi^{(D-1)/2}\Gamma(D/2+1/2)}\right]^{1/D} \tag{36}$$

is the ultrarelativistic thermal wavelength. Equations (30)–(35) represent the statistical properties of an ultrarelativistic q-deformed Bose gas and are just the results obtained in [20] with the case of $s = 1$ and $a = c$.

It should be pointed out that the definitions of the notation $[x]$, Hamiltonian $\hat{H}$ and mean value of the occupation number $f_{k,q}$ are not unique in the literature [12–15]. This may result in different distribution functions, and consequently, different dependences of the thermodynamic quantities of q-deformed systems on the parameter q. For example, it is found that, in the nonrelativistic limit, the expressions of the critical temperature and heat capacity in the present paper and [12, 20] are different from the corresponding expressions in [15].

3. Discussion

3.1. The condition of the occurrence of BEC

Using equation (15), one can obtain the condition whether BEC can occur at a nonzero temperature in a relativistic q-deformed Bose system or not. For the system of massive particles, i.e., $u_C = \beta_C mc^2 \neq 0$, we have $ju_C \gg 1$ for j greater than certain large value j_m. By using equation (21), equation (15) can be expressed as

$$N = \left(\frac{L}{\lambda_c}\right)^D \left(\frac{2\pi}{u_C}\right)^{D'-1} \left[\sum_{j=1}^{j_m-1} \frac{(1-q^{-2j})}{\ln q} \frac{\exp(ju_C)}{j^{D'}} K_{D'}(ju_C)\right] + \left(\frac{L}{\lambda_c}\right)^D \left(\frac{2\pi}{u_C}\right)^{D/2} \sum_{j=j_m}^{\infty} \frac{(1-q^{-2j})}{2\ln q} \frac{1}{j^{D/2+1}} \times \left[1 + \frac{4D'^2-1}{8}\left(\frac{1}{ju_C}\right) + \frac{16D'^4 - 40D'^2 + 9}{128}\left(\frac{1}{ju_C}\right)^2 + \cdots\right]. \tag{37}$$

When $q > 1$, the second term on the right-hand side in equation (37) is convergent, and hence $T_C \neq 0$ for any D. When $q \to 1$, however, equation (37) is reduced to

$$N = 2\left(\frac{L}{\lambda_c}\right)^D \left(\frac{2\pi}{u_C}\right)^{D'-1} \left[\sum_{j=1}^{j_m-1} \frac{\exp(ju_C)}{j^{D'-1}} K_{D'}(ju_C)\right] + \left(\frac{L}{\lambda_c}\right)^D \left(\frac{2\pi}{u_C}\right)^{D/2} \sum_{j=j_m}^{\infty} \frac{1}{j^{D/2-1}} \times \left[1 + \frac{4D'^2-1}{8}\left(\frac{1}{ju_C}\right) + \frac{16D'^4 - 40D'^2 + 9}{128}\left(\frac{1}{ju_C}\right)^2 + \cdots\right]. \tag{38}$$

It is found that the second term on the right-hand side in equation (38) is convergent, and hence $T_C \neq 0$ only for $D > 2$.

For the system of massless particles, i.e., $u_C = \beta_C mc^2 \to 0$, the critical temperature is given by equation (31). When $q > 1$, $\zeta_q(D+1)$ is convergent and $T_C \neq 0$ for any D. When $q \to 1$, however, $\zeta_q(D+1) \to \zeta(D)$ and $T_C \neq 0$ only for $D > 1$.

To sum up, BEC can occur in any dimensional space for a relativistic q-deformed Bose system of both massive and massless particles. The result is different from that of an ordinary Bose system. It has been proved that BEC can occur only in the case of $D > 2$ and $D > 1$ for the ordinary Bose system of massive and massless particles, respectively [24–27]. This shows that q-deformation significantly alters the characteristics of BEC for the low-dimensional systems.

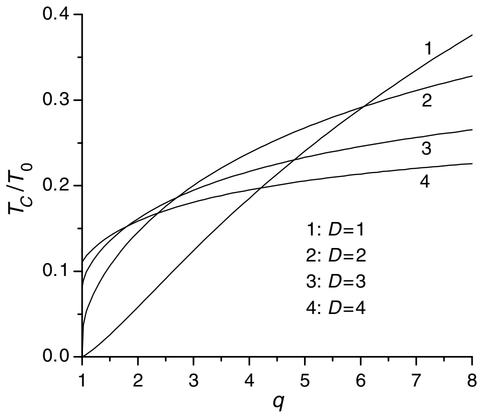

Figure 1. The critical temperature of q-bosons as a function of the parameter q in the cases of different spatial dimensionalities.

3.2. The discontinuity of the heat capacity at the critical temperature

Using equation (20) and the similar method mentioned above, we can obtain the condition of the discontinuity of heat capacity at the critical temperature, which is proved to be $D > 2$ and $D > 1$ for the system of massive and massless particles, respectively. The result gives rise to another difference between the q-deformed and ordinary Bose systems. It has been proved that for the ordinary Bose system of massive and massless particles, the heat capacity at the critical temperature is discontinuous when $D > 4$ and $D > 2$, respectively [24–27].

3.3. The effect of q-deformation on the critical temperature

It is seen from equation (15) that the critical temperature T_C is dependent on the density of particles $\rho = N/L^D$, the rest mass of a particle m, the dimensionality of space D and the deformation parameter q. Figure 1 shows the curves of the scaled critical temperature T_C/T_0 varying with q for different D, where $m = m_u = 1$ (au), $\rho_1 \equiv \rho^{1/D} = 1/\lambda_c$ and $T_0 \equiv m_u c^2/k_B$ are chosen. It can be seen from figure 1 that q-deformation results in the increase of the critical temperature for an any dimensional q-deformed system. The increasing rates of T_C/T_0 with q, however, are different for different D. For the low-dimensional systems, such as $D = 1$ and 2, T_C/T_0 increases quickly with q and the critical temperature may be much higher than that of the corresponding undeformed systems ($q = 1$) when q is large. In contrast, for the high-dimensional systems, such as $D = 4$, the increase of T_C/T_0 with q is very slow, so that the critical temperature of the q-deformed system may be very close to that of the undeformed system. Another characteristic appearing in figure 1 is that the critical temperature of q-bosons with a large parameter q decreases with the spatial dimensionality for the given one-dimensional density ρ_1. The result is in contrary to that of the ordinary bosons ($q = 1$). It is well known that critical temperature of the ordinary bosons increases with the spatial dimensionality for the given ρ_1.

Figure 2 shows the dependence of the scaled critical temperature T_C/T_0 on the scaled rest mass of a particle m/m_u for $D = 3$ different parameter q. The effects of q-deformation on the critical temperature from the nonrelativistic case to the ultrarelativistic limit can be clearly seen in this figure.

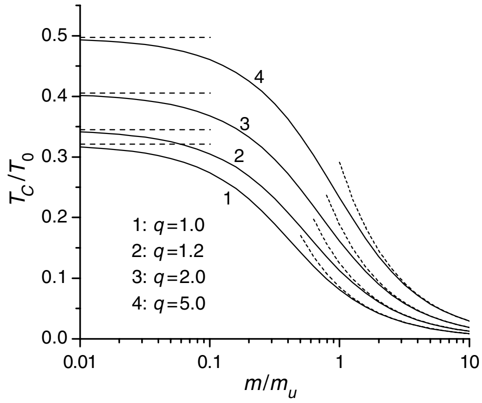

Figure 2. The critical temperature of q-bosons as a function of the rest mass of a particle from the nonrelativistic case to the ultrarelativistic limit.

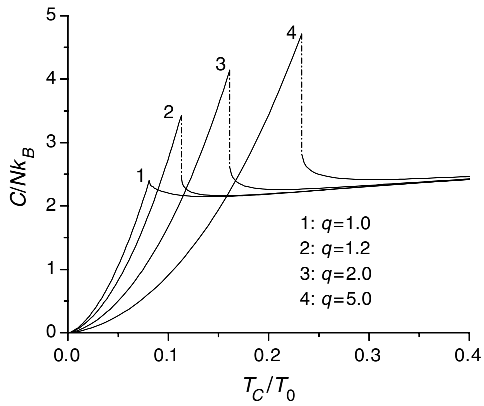

Figure 3. The curves of the heat capacity varying with temperature for the three-dimensional q-deformed Bose system in the cases of different parameter q.

3.4. The dependence of the heat capacity on the temperature

Using equations (11), (18) and (19), one can expound the dependence of the heat capacity on the temperature. Figure 3 shows the curves of the scaled heat capacity C/Nk_B varying with the scaled temperature T_C/T_0 for $m = m_u = 1$ (au), $\rho_1 = 1/\lambda_c$ and $D = 3$. It is seen that the heat capacity at the critical temperature is discontinuous when $q \neq 1$. The gap $\Delta C \equiv C_{T=T_C^-} - C_{T=T_C^+}$, which is determined by equation (20), decreases with the decrease of q and $\Delta C \to 0$ when $q \to 1$. At high temperatures, the heat capacities converge toward the same value as that of Boltzmann particles and are independent of q. This indicates that q-deformation is a pure quantum effect.

It should be noted in final that, although the properties of the q-deformed Bose system are only discussed, the results obtained in the present paper are valid as well for the ordinary Bose system as long as $q \to 1$ is set. For example, when $q \to 1$, the expressions for the critical temperature in the cases of nonrelativistic and ultrarelativistic limits, i.e., equations (23) and (31), are just the same as the corresponding results given in a recently published paper [28].

4. Conclusions

In summary, we have studied the low-temperature behavior of a relativistic q-deformed Bose gas in a D-dimensional space. The effects of q-deformation on the characteristics of BEC are discussed and some important conclusions are obtained as follows: (1) BEC can occur in an any dimensional system of both massive and massless q-bosons. (2) The condition that the heat capacity at the critical point is discontinuous is $D > 2$ and $D > 1$ for the relativistic q-deformed Bose systems of massive and massless q-bosons, respectively. (3) The q-deformation generally results in the increase of the critical temperature. The effect is more significant in the low-dimensional systems than in the high-dimensional systems. (4) The q-deformation increases the gap of the heat capacity at the critical temperature. (5) At high temperatures, the properties of q-bosons will become the same as those of Boltzmann particles, which is independent of q, and therefore q-deformation is a pure quantum effect.

Acknowledgment

This work has been supported by the Research Foundation of the Ministry of Education (no 20050384005), People's Republic of China.

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